A Novel Sub-Nyquist Sampling of Sparse Wideband Signals

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***Abstract*—We propose a Sub-Nyquist sampling scheme using only one mixer, one filter bank with M subfilters and M low rate ADCs. Compared to the MWC (Modulated Wideband Converter) system in [1], the proposed sampling system has several complexity advantages. First, due to the use of only one mixer, the hardware complexity for analog circuits is significantly reduced. Moreover, the system only needs one PN sequence and thus the search complexity for the best sequence is greatly reduced. The use of the best sequence enables the proposed system to achieve a smaller mean square error (MSE) between the original and the reconstructed signals than that of the MWC system. Simulation results are provided to show the performance superiority of the proposed system in terms of MSE and recovery percentage.**

Index Terms — Compressive sensing (CS), sub-Nyquist sampling, spectrum blind reconstruction, modulated wideband converter (MWC), random demodulator (RD).

I. Introduction

Compressive sensing (CS) is proposed recently for the acquisition of sparse signals using a sampling rate significantly lower than Nyqiust rate. The reconstructed signals in compressive sensing can be treated as a solution of the 1-norm optimization problem [2]. Moreover, CS has been particularly influential in contributing insights into the design of wideband receivers, see *e.g.* [1], [3] and [4].

Using a wavelet edge detector with the derivative of the power spectrum density, and matrix algebraic operations, the authors in [3] presented a compressive wideband spectrum sensing scheme for cognitive radios. Also, a method called random demodulator (RD) was recently introduced as a wideband receiver for reducing the sampling rate below Nyquist rate [4]. This approach, however, should implement the hardware with a very high rate pseudo-random mixer which leads to huge computational complexity of signal reconstructions for DSP processors. Thus, a scheme called the modulated wideband converter (MWC) is proposed in [1] to overcome this issue. The MWC mitigates the need of a high bandwidth and hence reduces the complexity of designs. More specifically, the MWC design consists of an analog frond-end with several branches. In each branch, the input signal is multiplied by a pseudo-random periodic waveform, lowpass filtered, and then sampled by a low rate ADC. However, this MWC method needs several branches that consist of many sets of mixers, lowpass filters and ADCs, and thus the complexity of the hardware is still high. Moreover, since there are several mixers in MWC, the system needs to determine several sets of the most suitable PN sequences for these mixers. Although this can be done off-line, it may still be a difficult task. For instance, if the number of branches is 35 and the PN sequences are of length 511. Then, the search effort for the best 35 PN sequences is to choose the 35 best sequences from $2^9 + 1$ candidates. Therefore the computational complexity is usually prohibitive in this case.

In this paper we propose a novel sub-Nyquist sampling scheme using only one mixer, one filter bank with M sub-filters and M low rate ADCs. We also apply quadrature-mirror-filter (QMF) design method for implementing the filter bank which is commonly used in multi-resolution systems. A pair of filters split the input signal into two bands, namely high-passed and low-passed signals, and result a critically sampled two-channel representation of the original signal. The process will be carried further until the filter bank divides the signal into M subbands. QMF is a mature technique using analog or digital filter designs with CMOS circuits. Paper [5] designed an analog QMF, a tree-structured filter bank realized in 0.35 μm CMOS technology, with switched capacitor FIR filters. Compared to the MWC method that uses one low filter on M paths, the proposed structure requires a filter bank. Except this minor disadvantage, the proposed front-end has several implementation advantages: First, due to the use of only one PN generator and one mixer, the complexity of analog circuit design and the effort for timing synchronization between M paths are significantly reduced. Moreover, we find the compressive sensing (CS) matrix of the proposed structure is a Toeplitz matrix. As a result, the matrix computations can also be dramatically simplified. Furthermore, the system only needs one PN sequence and thus the search complexity for the best sequence is greatly reduced. The use of the best sequence enables the proposed system to slightly outperform the MWC system in terms of MSE.

Notations. Let $s(t)$, $s[n]$, $S(f)$ and $\mathbf{s}(f)$ denote the continuous-time signal, discrete-time signal, Fourier transform of $s(t)$ and vector that depends on a continuous parameter f, respectively. The notations $\mathbf{M}_i$, $\mathbf{M}^*$, $\mathbf{M}^H$ and $\mathbf{M}^\dagger$ respectively represent the ith row of $\mathbf{M}$, conjugate, conjugate transpose, and pseudo-inverse of matrix $\mathbf{M}$. $\lceil\cdot\rceil$ denotes the ceil operator.

II. SYSTEM MODEL of Proposed Scheme

This paper considers wideband signals, which has sparsity in frequency domain. Let the Nyquist frequency of the signal be W. We assume there are N narrowband sparses with each bandwidth of sparse is less than B. Due to the sparsity of the signal, $NB \ll W$, and such sparse signal is also called "multiband" signal. The frequency spectrum of a multiband signal is shown in Fig. 1(a) for instance.

The idea of proposed scheme is the multiband input signal $s(t)$ is multiplied by a mixing function $p(t)$ and the frequency spectrum $P(f)$ of a multiband signal is shown in Fig. 1(b) for instance. The frequency spectrum of the modulated signal, *e.g.* a PN sequence, is the multiband input signal $S(f)$ convoluted by the mixing function $P(f)$, if the modulated signal is passed into an ideal lowpass filter. Thus we can obtain a baseband signal as shown in Fig. 1(c) for instance, and this frequency spectrum is one of the branch output of the MWC system. To capture sufficient signal information, the MWC system need to use several different PN sequences, and this requires several mixers. As mentioned in Introduction, the design effort increases as the number of mixers increases. To avoid using multiple mixers, we notice that the frequency information of the modulated signal is distributed in all frequency spectrum likes that shown in Fig. 1(d). Therefore we can use a filter bank to capture all the frequency information and then reconstruct the sparse signal. By doing this, we only need one mixer.

The proposed Sub-Nyquist sampling scheme is shown in Fig. 2, where there are one mixer, M subfilters and M ADCs. Note that if M is sufficient large, the proposed scheme does not need prior

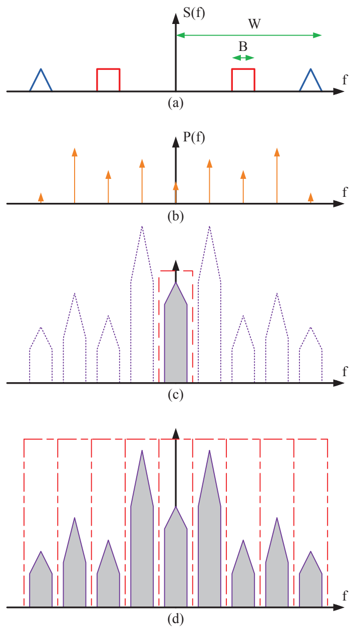

Fig. 1: Frequency spectrum illustration. (a) Frequency spectrum of a multiband signal, (b) Frequency spectrum of a mixing function, (c) Frequency information of MWC system after lowpass filtering (in one of the branches), (d) Frequency information of the proposed system by using filter bank.

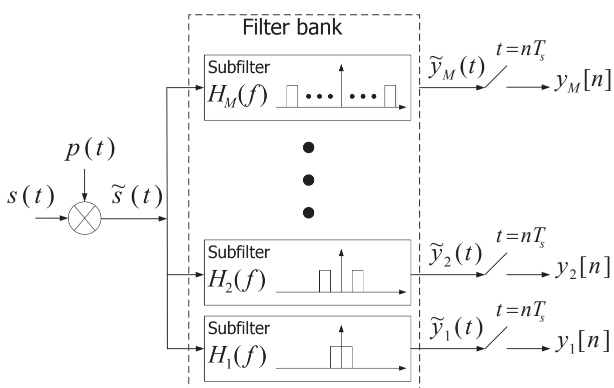

Fig. 2: A block diagram of the proposed sampling system which consists of one mixer, a filter bank with M subfilters and ADCs.

knowledge of the sparse locations to reconstruct the original signal. Later we will show the relationship between M and N for a robust reconstruction. As shown in Fig. 2, the input signal $s(t)$ is multiplied by a mixing function $p(t)$, which is T_p-periodic. Assume $1/T_p \geq B$, after mixing, the modulated signal is passed into the M-branch filter bank and ADCs. Assume the sampling rate of ADCs be $1/T_s$. The modulated signal $\tilde{s}(t)$ is truncated by the subfilters, which have different central frequencies and the same cutoff frequency $1/(2T_s)$. Let the frequency response of the mth subfilter be $H_m(f)$. In general, the sampling frequency of each branch is low enough so that the ADCs can be designed with less technical efforts and lower power consumptions. The four design parameters to be determined are M, T_p, $H_m(f)$ and $p(t)$ as described in the following section.

III. Problem Formulations

In this section, we explain the properties of the proposed scheme in frequency domain. For the mixing function $p(t)$, it can be chosen as a piecewise continuous-time function that takes values ± 1 with equal probability for each of the P equal time intervals, *i.e.*

$$p(t) = \beta_k,\ k\frac{T_p}{P} \leq t \leq (k+1)\frac{T_p}{P},\ 0 \leq k \leq P-1, \quad (1)$$

where $\beta_k \in \{+1, -1\}$, $p(t) = p(t + nT_p)$ for every $n \in \mathbb{Z}$ and $1/T_p = f_p$. We can choose a periodic pseudo-random sequence as the mixing function $p(t)$ for the proposed scheme. Since the mixing function $p(t)$ is T_p-periodic, it has a Fourier expansion given by

$$p(t) = \sum_{i=-\infty}^{\infty} c_i e^{j\frac{2\pi}{T_p}it}, \quad (2)$$

The Fourier transform of the modulated signal $\tilde{s}(t) = s(t)p(t)$ and can be shown to be [1]

$$\tilde{S}(f) = \sum_{i=-\infty}^{\infty} c_i S(f - if_p). \quad (3)$$

Next let us define $H_m(f)$. In this paper, we assume the number of subfilters is arbitrary. For description convenience, we describe the filter band in the positive frequency band that the first subfilter $H_1(f)$ is an ideal lowpass filter and the other subfilters are ideal bandpass filters defined as

$$H_m(f) = \begin{cases} 1 & \frac{(m-1)}{2}f_s \leq f \leq \frac{m}{2}f_s \\ 0 & \text{otherwise} \end{cases},\ 1 \leq m \leq M, \quad (4)$$

where $1/T_s = f_s$ is the sampling frequency of the ADCs. For the proposed system, the sampling rate of the ADCs satisfies $f_s = qf_p$ with $q \in \mathbb{N}$, where q is the sampling factor.

Now let us look at the sampling signal $y_m[n]$ in Fig. 2. From (3), originally the modulated signal $\tilde{s}(t)$ is an infinite linear combination of f_p-shifted copies of $S(f)$. After passing to the subfilter $H_m(f)$, however, it becomes a finite linear combination, thus $y_m[n]$ can be expressed as

$$Y_m(f) = \sum_{i=-\mathcal{I}}^{\mathcal{I}} c_i S(f - if_p),\ f \in \left[\frac{(m-1)}{2}f_s, \frac{m}{2}f_s\right], \quad (5)$$

where the value of $\mathcal{I}$ is calculated by

$$\mathcal{I} = \left\lceil \frac{W + f_s}{2f_p} \right\rceil - 1. \quad (6)$$

Let us describe the signal processing between signal $\tilde{y}_m(t)$ and $y_m[n]$ in Fig. 2. For $\tilde{y}_1(t)$, which is a baseband signal to be filtered by the lowpass filter $H_1(f)$, the output $y_m[n]$ is exactly the sampled signal from the ADC. For $\tilde{y}_m(t)$, $m \neq 1$, however, which are bandpass signals and their modulated frequency are higher than the sampling rate of the ADCs, we need to apply the bandpass theory and let $f_s = qf_p$, $q \in \mathbb{N}$, to avoid aliasing [6]. In this case, each of the resulting signal $\tilde{y}_m(t)$, $m \neq 1$ is not aliasing, and may be regarded as an interpolated signal which has several copies in frequency domain. By passing $\tilde{y}_m[n]$, $m \neq 1$ into a digital ideal filter with passband in $[0\ f_s/2]$, or a proper downsampling process (see *e.g.* [6]), the resulting signal $z_m[n]$, $m \neq 1$ is a baseband signal with information contained in $[0\ , +f_s/2]$. For analysis convenience, we let $f_s = f_p$ and the DTFT of $z_m[n]$ be $Z_m(f)$, which can be expressed as

$$Z_m(f) = \sum_{i=-\mathcal{I}}^{\mathcal{I}} c_i S(f - (i-m)f_s),\ f \in \left[0, +\frac{f_s}{2}\right] \quad (7)$$

The equation in (7) can be represented by a matrix form, *i.e.*

$$Z_m(f) = \boldsymbol{\phi}_1 \mathbf{x}_m(f) \tag{8}$$

where

$$\boldsymbol{\phi}_1 = \left[\begin{array}{ccccc} c_{-\mathcal{I}} & \cdots & c_0 & \cdots & c_{\mathcal{I}} \end{array} \right], \tag{9}$$

and

$$\mathbf{x}_m(f) = \left[\begin{array}{cc} S(f+(\mathcal{I}-m+1)f_p) \cdots S(f-\mathcal{I}f_p) & \mathbf{0}^T \end{array} \right]^T, \tag{10}$$

is a $(2\mathcal{I}+1) \times 1$ column vector with its last $m-1$ elements being zeros. It is equivalent to rewriting (8) as

$$Z_m(f) = \boldsymbol{\phi}_1 \mathbf{x}_m(f) = \boldsymbol{\phi}_m \mathbf{x}_1(f), \tag{11}$$

where

$$\boldsymbol{\phi}_m = \left[\begin{array}{cc} \mathbf{0}^T & c_{-\mathcal{I}} \cdots c_{\mathcal{I}-m+1} \end{array} \right], \tag{12}$$

with its first $m-1$ elements being zeros. Let us collect the M $\boldsymbol{\phi}_m$ vectors from the M branches and define the matrix $\boldsymbol{\Phi}$ as

$$\boldsymbol{\Phi} = \left[\begin{array}{c} \boldsymbol{\phi}_1^T \ \boldsymbol{\phi}_2^T \cdots \boldsymbol{\phi}_M^T \end{array} \right]^T.$$

Then, (11) can be rewritten in a matrix from given by

$$\mathbf{z}(f) = \boldsymbol{\Phi}\mathbf{x}_1(f), \tag{13}$$

where

$$\mathbf{z}(f) = \left[\begin{array}{ccc} Z_1(f) & \cdots & Z_M(f) \end{array} \right]^T,$$

and $\boldsymbol{\Phi} \in \mathbb{C}^{M\times(2\mathcal{I}+1)}$ is referred as the CS (compressive sensing) matrix. We find that the CS matrix of the proposed scheme is a Toeplitz matrix. For instance, let $\mathcal{I} = 3$ and $M = 3$, then the matrix $\boldsymbol{\Phi}$ can be expressed as

$$\boldsymbol{\Phi} = \left[\begin{array}{ccccccc} c_{-3} & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & c_3 \\ 0 & c_{-3} & c_{-2} & c_{-1} & c_0 & c_1 & c_2 \\ 0 & 0 & c_{-3} & c_{-2} & c_{-1} & c_0 & c_1 \end{array} \right].$$

About the coefficients c_i in matrix $\boldsymbol{\Phi}$, they can be obtained by performing the Fourier transform of the β_k, *i.e.* [1]

$$c_i = \frac{1}{P}\sum_{k=0}^{P-1} \beta_k e^{-j\frac{2\pi}{P}ik}, \quad \text{for} \ -\mathcal{I} \le i \le \mathcal{I}.$$

Thanks to the Toeplitz structure of matrix $\boldsymbol{\Phi}$, the storage requirement is only $1/(2M)$ of that in the MWC scheme [1]. For instance, later in the simulation results, we would like to handle a 10 GHz signals by $M = 35$ branches. In this case, the proposed scheme requires a memory size of around 100×1, whereas the MWC scheme demands the memory size of 100×35.

A. Relationship Between f_s, f_p and B

Let us discuss the relationship between the sampling rate f_s, the chip rate f_p and the subband signal bandwidth B. First, to ensure satisfactory reconstruction. We demand $f_p \ge B$. The reason is that from (3), the two copies signals $S(f - if_p)$ and $S(f - (i+1)f_p)$ do not overlap too serious when $f_p \ge B$. Without this condition, the copies of $S(f - if_p)$ for different i are seriously overlapping and perfectly reconstruction is not possible even if the system is in a noise-free environment. That is, this condition helps preserving the sparsity property of the signal $\mathbf{x_1}(f)$ defined in (10) and hence there are at most N nonzero elements in $\mathbf{x_1}(f)$. Second, we should let $f_s = qf_p \ge B$, $q \in \mathbb{N}$. That q must be an integer is to satisfy the condition for bandpass theory discussed in Sec. 2. Moreover, for large q, the number of branches decreases and thus the numbers of ADCs and the subfilters also decrease.

B. Signal Reconstruction

Next let us discuss how to reconstruct the compressed signal: the authors in [1] proposed a method, called continuous to finite (CTF) to reconstruct $\mathbf{x}_1(f)$ by solving a compressive sensing (CS) problem. Let $\mathbf{z}[n]$ be the IDTFT of $\mathbf{z}(f)$. In order to use CTF, we need to obtain a frame $\mathbf{V}$ for the measurement set, Such a frame can be obtained by computing

$$\mathbf{R} = \int_{f\in f_s} \mathbf{z}(f)\mathbf{z}^H(f)df = \sum_{n=-\infty}^{\infty} \mathbf{z}[n]\mathbf{z}^H[n].$$

Then perform the matrix decomposition (EVD) of $\mathbf{R}$, *i.e.* $\mathbf{R} = \mathbf{V}\mathbf{V}^H$. The matrix $\mathbf{V}$ can be expressed as

$$\mathbf{V} = \boldsymbol{\Phi}\mathbf{X},$$

where the joint sparsity of $\mathbf{X}$ equals to the sparsity of $\mathbf{x}_1(f)$ [7] and solving $\mathbf{X}$ is referred to as a multiple measurement vectors (MMV) problem [8]. By observing the sparse location of $\mathbf{X}$, we know which columns of $\boldsymbol{\Phi}$ should be used to reconstruct the signal, denoted the matrix consisting of these columns by $\boldsymbol{\Phi}_\lambda$. Then the signal $\mathbf{x}_\lambda[n]$ can be obtained by

$$\mathbf{x}_\lambda[n] = \boldsymbol{\Phi}_\lambda^\dagger \mathbf{z}[n].$$

Once having $\mathbf{x}_\lambda[n]$, we can reconstruct the original signal by modulating the elements of $\mathbf{x}_\lambda[n]$ to the corresponding frequencies and then can combine the modulated signals and if MMV is used, the authors in [1] have shown the relationship between M and N for a satisfactory reconstruction should be

$$M \approx 4N\log(P/2N). \tag{14}$$

IV. SIMULATION RESULTS

We conducted computer simulations to validate the performance of the proposed scheme. According to the parameter settings in [1], the system conditions are as follows: A radio signal consists of three pairs of bands ($N = 6$), with each bandwidth $B = 50$ MHz, and this radio signal is formulated as

$$x(t) = \sum_{i=1}^{3} \sqrt{E_i B}\,\mathrm{sinc}(B(t-\tau_i))\cos 2\pi f_i(t-\tau_i). \tag{15}$$

where $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. The carrier frequencies f_i are chosen randomly in $[-W/2, W/2]$ with $W = 10$ GHz, also the energy and time offset are $E_i = \{1, 2, 3\}$ and $\tau_i = \{0.4, 0.7, 0.2\}\,\mu$s, respectively. For the mixing function p(t), since Gold sequences are claimed in MWC scheme [9] having the best performance compared with other binary sequence families, we use Gold sequence of $2^9 - 1$ in our simulations. Fig. 3 validates the proposed scheme can reconstruct the original spectrum accurately by using parameters of $P = 200$, $f_s = f_p = 50$ MHz, $M = 25$ and signal-to-noise ratio (SNR) $= 20$ dB. We also compute the normalized MSE between the reconstructed signal and the original signal for the proposed scheme, and then compared the MSE performance with MWC scheme [1]. The normalized MSE is defined as $\|\hat{x}(t) - x(t)\|^2/\|x(t)\|^2$. For the MWC method, since it needs M mixers to implement the system, we randomly choose M Gold sequences to construct CS matrix $\boldsymbol{\Phi}$, because the computational effort to search exhaustively the best M sequences from 2^9+1 Gold sequences may be somewhat impractical. For the proposed system, since we only need one mixer, the search for the best Gold sequence is much easier, and this advantage reflects on the better MSE performance as shown in Fig. 4. Moreover, Fig. 5 shows that the MSE performance is obtained by a chosen Gold sequence is better than a randomized Gold sequence. Note that the chosen Gold sequence is obtained by exhaustive search. This result

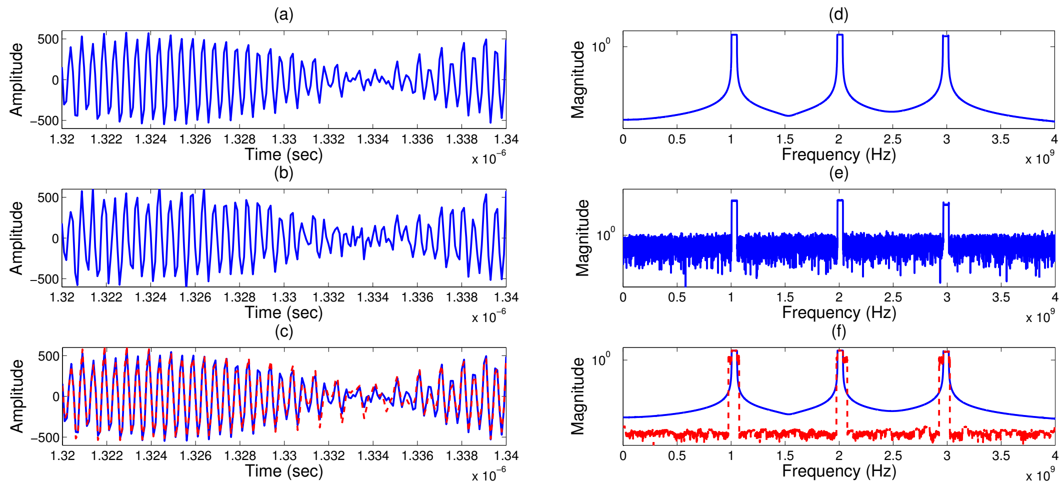

Fig. 3: Reconstruction of multiband signal. (a) Time domain waveform of the original signal, (b) Time domain waveform of the noisy signal, (c) Time domain waveform of the original and reconstructed signal (dotted line), (d) Spectrum of the original signal, (e) Spectrum of the noisy signal, (f) Spectrum of the original and reconstructed signal (dotted line).

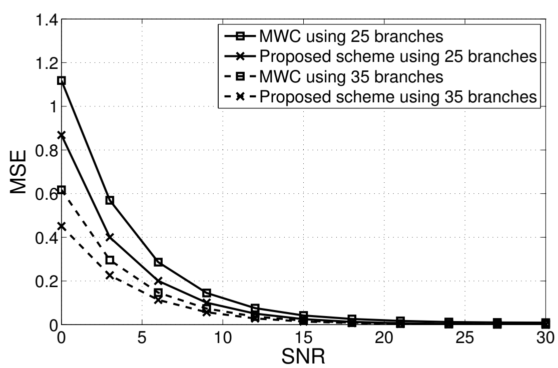

Fig. 4: MSE performance with different M

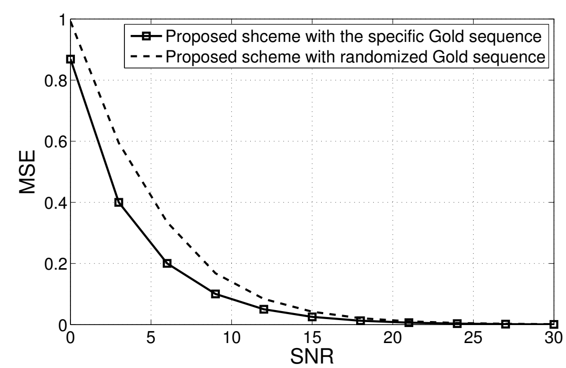

Fig. 5: MSE performance with different sequence

presents the proposed system improves the MSE performance by carefully choosing a sequence because only one sequence is needed. On the other hand, MWC system needs several sequences and thus the required exhaustive search may cost a huge computational complexity. Finally, Fig. 6 shows that the percentage of correct support recoveries for various branches vs. SNRs. This result presents that at high SNR regime the correct support recovery is achieved when the proposed scheme uses the number of branches $M \geq 35$. Further, the ratio of the Nyquist rate and the sampling rate of proposed scheme with 35 branches is equal to 0.15. Besides, the describes the relationship between Fig. 6 and $4N \log(P/2N) \approx 30$ of equation (14). The results confirm the proposed sub-Nyquist sampling scheme works well and is a promising structure for compressive sensing of sparse wideband signals.

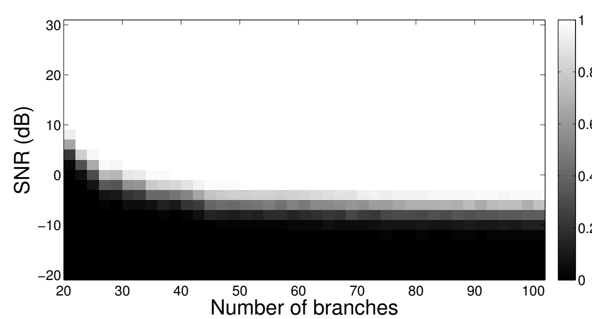

Fig. 6: The percentage of the correct support set recovery as a function of the number of branches and the SNR.

V. CONCLUSION

In this paper, we presented a novel Sub-Nyquist sampling system. The scheme significantly reduces the design complexity because it only needs one mixing function with a Toeplitz-structured CS matrix, therefore using one mixing function can decrease designed complexity of CS matrix that property of the CS matrix affect performance of reconstruction strongly. From the simulation results, the proposed system slightly outperforms the MWC system in terms of MSE, which reflects the advantage of using of the best PN sequence on the proposed system.

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